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A. H. Zakari · A. Ghareeb · Y. C. Kim

Fuzzy $k_\circ$-preproximities related to fuzzy closure spaces

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Abstract The aim of this paper is to define the concept of fuzzy k_0-preproximity and show how a fuzzy closure space is induced by a fuzzy k_0-preproximity and vice versa. Also, we introduce the notion of fuzzy k_0-preproximal neighborhood system.

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الملخص

هدف هذه الورقة هو تعريف مفهوم قبل القُرْب- k_0 المُشَوَّش وإثبات أن فضاء الإغلاق المُشَوَّش ينجم عن قبل قُرْب- k_0 مُشَوَّش والعكس بالعكس. نقدم أيضاً مفهوم نظام جوار قبل قُرْب- k_0 مُشَوَّش.

1 Introduction

Zadeh [12] introduced the theory of fuzzy sets. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. The study of fuzzy topology structures started with the introduction of the concept of fuzzy topology in the pioneering paper of Chang [3]. Kubiak [9] and Šostak [11] introduced a generalization of Chang's fuzzy topology depending on fuzzy sets. The resulting structure is given the name "L-fuzzy topology" where L is any appropriate lattice. Chattopadhyay and Samanta [4] defined the fuzzy closure operator and established some characteristic properties of this operator.

Efremovič [5] introduced the proximity relation. Han [6] introduced the $k_\circ$-proximity space as a generalization of the Efremovič-proximity space. Katsaras [8] defined fuzzy proximities, on the base of the axioms suggested by Efremovič. Park [10] generalized the concept of the fuzzy proximity, which was called by a fuzzy k-proximity. The concept of fuzzy proximity has met the attention of many researchers (see Zahran et al. [13,14], Abbas and Abd-Allah [1], Çetkin and Aygün [2]).

Topology is one of the core concepts in Geospatial Information System (GIS). In order to fully define and model fuzzy spatial objects such as land covers, it is necessary to investigate their fuzzy topological

A. H. Zakari (✉)
Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia
E-mail: d_ahz@hotmail.com

A. Ghareeb
Department of Mathematics, College of Science in Al-Zulfi, Majmaah University, Al-Zulfi, Saudi Arabia
E-mail: nasserfuzt@aim.com

Y. C. Kim
Department of Mathematics, Gangneung-Wonju National University, Gangneung, Gangwondo 210-702, Korea
E-mail: yck@gwnu.ac.kr

relationships. Furthermore, Topology and its theoretical applications have a wide range of scientific applications in robotics, artificial intelligence. Also, the lattice can be used for the GIS modeling which can reduce the complexity of some quires (see Kains et al. [7]). Topological spaces have strong close relation between them and closure spaces which, at the same time, have important relations with proximity spaces.

In this paper, we introduce the notion of fuzzy $k_\circ$-preproximity and investigate some of its properties. Also, we study the relationships between fuzzy closure spaces, fuzzy $k_\circ$-preproximity, fuzzy cotopology and fuzzy neighborhood system.

2 Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = I - \{0\}$ and $I_1 = I - \{1\}$. A fuzzy point x_t is a fuzzy set on X defined by $x_t(x) = t$ and $x_t(y) = 0$ for all $y \neq x$, where $t \in I_0$. Then $Pt(X) = \{x_t \in I^X : x \in X,\ t \in I_0\}$ is the set of all fuzzy points of I^X. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The complement μ' of fuzzy set μ in X is defined by $\mu'(x) = 1 - \mu(x)$ for each $x \in X$.

Definition 2.1 (Kubiak [9], Šostak [11]) A mapping $\mathcal{T} : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1$.
(O2) $\mathcal{T}(\lambda \wedge \mu) \geq \mathcal{T}(\lambda) \wedge \mathcal{T}(\mu)$ for any $\lambda, \mu \in I^X$.
(O3) $\mathcal{T}(\bigvee_{k\in\Gamma} \lambda_k) \geq \bigwedge_{k\in\Gamma} \mathcal{T}(\lambda_k)$ for any $\{\lambda_k\}_{k\in\Gamma} \subset I^X$.

The pair $(X, \mathcal{T})$ is called a fuzzy topological space. Let $\mathcal{T} : I^X \to I$ be a fuzzy topology on X. Define the function $\mathcal{F} : I^X \to I$ such that $\mathcal{F}(\lambda) = \mathcal{T}(\lambda')$, then $\mathcal{F}$ is called a fuzzy cotopology on X. Clearly, $\mathcal{F}$ satisfies the following conditions:

(F1) $\mathcal{F}(\underline{0}) = \mathcal{F}(\underline{1}) = 1$.
(F2) $\mathcal{F}(\lambda_1 \vee \lambda_2) \geq \mathcal{F}(\lambda_1) \wedge \mathcal{F}(\lambda_2)$ for any $\lambda_1, \lambda_2 \in I^X$.
(F3) $\mathcal{F}(\bigwedge_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \mathcal{F}(\lambda_i)$ for any $\{\lambda_i\}_{i\in\Gamma} \subset I^X$.

Definition 2.2 (Chattopadhyay and Samanta [4]) A fuzzy closure space is an ordered pair $(X, \mathcal{C})$, where $\mathcal{C} : I^X \times I_0 \to I^X$ is a function satisfying the following conditions:

(C1) $\mathcal{C}(\underline{0}, r) = \underline{0}$ for all $r \in I_0$.
(C2) $\lambda \leq \mathcal{C}(\lambda, r)$ for all $\lambda \in I^X$, $r \in I_0$.
(C3) If $\lambda \leq \mu$ and $r \leq s$, then $\mathcal{C}(\lambda, r) \leq \mathcal{C}(\mu, s)$ for all $\lambda, \mu \in I^X$ and $r, s \in I_0$.
(C4) $\mathcal{C}(\lambda \vee \mu, r) = \mathcal{C}(\lambda, r) \vee \mathcal{C}(\mu, r)$ for all $\lambda, \mu \in I^X$ and $r \in I_0$.

3 Fuzzy $k_\circ$-preproximities

In this section, we introduce the concept of fuzzy $k_\circ$-preproximity and some of the definitions associated with it. Also, we generate fuzzy $k_\circ$-preproximity space from fuzzy closure space and vice versa.

Definition 3.1 A function $\delta : Pt(X) \times I^X \to I$ is called a fuzzy $k_\circ$-preproximity on X if it satisfies the following conditions:

(K1) $\delta(x_r, \underline{0}) = 0$ for all $r \in I_0$.
(K2) If $\lambda \leq \mu$ and $r \leq s$, we have $\delta(x_r, \lambda) \leq \delta(x_s, \mu)$ for each $r, s \in I_0$.
(K3) If $\delta(x_r, \mu) \neq 1$, then $x_r \in \mu'$.
(K4) $\delta(x_r, \rho_1) \vee \delta(x_s, \rho_2) = \delta(x_{r\wedge s}, \rho_1 \vee \rho_2)$.

The pair (X, δ) is called a fuzzy k_0-preproximity space. A fuzzy $k_\circ$-preproximity space is called principal provided that

(K5) $\delta(x_r, \bigvee_{j\in J} \lambda_j) \leq \bigvee_{j\in J} \delta(x_r, \lambda_j)$.

Let δ_1 and δ_2 be two fuzzy $k_\circ$-preproximities on X. We say that δ_1 is finer than δ_2 (δ_2 is coarser than δ_1), and write $\delta_1 \leq \delta_2$, iff $\delta_1(x_r, \lambda) \leq \delta_2(x_r, \lambda)$ for each $x_r \in Pt(X)$ and $\lambda \in I^X$.

Example 3.2 Let $\delta : Pt(X) \times I^X \to I$ defined as:

$$\delta(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{4}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.3}, r \in (0, 0.7]; \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.4}, r \in (0, 0.6]; \\ 1, & \text{otherwise.} \end{cases}$$

Then δ is a fuzzy $k_\circ$-preproximity.

Theorem 3.3 *Let (X, δ) be a fuzzy $k_\circ$-preproximity space. Define a function $\mathcal{C}_\delta : I^X \times I_0 \to I^X$, for each $x \in X$, by:*

$$\mathcal{C}_\delta(\lambda, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, s \in I_0; \\ \bigwedge\{x'_r \in I^X : \delta(x_r, \lambda) < s'\}, & \text{if } \delta(x_r, \lambda) < s'; \\ \underline{1}, & \text{otherwise.} \end{cases}$$

Then the pair $(X, \mathcal{C}_\delta)$ is a fuzzy closure space.

Proof

(C1) Clear.
(C2) Let $\delta(x_r, \lambda) < s'$, $\lambda \neq \underline{0}$. Using (K2), $\lambda \leq x'_r$, since $\delta(x_r, \lambda) \neq 1$. Thus, $\mathcal{C}_\delta(\lambda, s) \geq \lambda$ for all $\lambda \in I^X$, $s \in L_0$.
(C3) Let $\lambda \leq \mu$ and $s \leq q$. Then $\delta(x_r, \lambda) \leq \delta(x_r, \mu)$. Let $\delta(x_r, \lambda) < s'$ and $\delta(x_r, \mu) < q'$. By the definition of $\mathcal{C}_\delta$, we have

$$\begin{aligned} \mathcal{C}_\delta(\mu, q) &= \bigwedge\{x'_r \in I^X : \delta(x_r, \mu) < q'\} \\ &\geq \bigwedge\{x'_r \in I^X : \delta(x_r, \lambda) < q'\} \\ &\geq \bigwedge\{x'_r \in I^X : \delta(x_r, \lambda) < s'\} \\ &= \mathcal{C}_\delta(\lambda, s). \end{aligned}$$

(C4) From (K2) and (K4), we have

$$\begin{aligned} \mathcal{C}_\delta(\lambda, s) \vee \mathcal{C}_\delta(\mu, s) &= \big(\bigwedge\{x'_r \in I^X : \delta(x_r\lambda) < s'\}\big) \vee \big(\bigwedge\{x'_t \in I^X : \delta(x_t, \mu) < s'\}\big) \\ &= \bigwedge\{(x_r \wedge x_t)' \in I^X : \delta(x_r, \lambda) < s', \delta(x_t, \mu) < s'\} \\ &\geq \bigwedge\{(x_r \wedge x_t)' \in I^X : \delta(x_r, \lambda) \vee \delta(x_t, \mu) < s'\} \\ &\geq \bigwedge\{x'_k \in I^X : \delta(x_k, \lambda \vee \mu) < s'\} \\ &= \mathcal{C}_\delta(\lambda \vee \mu, s). \end{aligned}$$

□

The following example shows that we can get a fuzzy closure space from a fuzzy $k_\circ$-preproximity space.

Example 3.4 Let $\delta_1, \delta_2 : Pt(X) \times I^X \to I$ defined as:

$$\delta_1(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\delta_2(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{4}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\delta_1(x_r, \lambda)$ and $\delta_2(x_r, \lambda)$ are fuzzy $k_\circ$-preproximities. We will create fuzzy closure operators from the above fuzzy $k_\circ$-preproximities as follows:

$$\mathcal{C}_{\delta_1}(\lambda, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, s \in (0, 1); \\ x'_{0.9}, & \text{if } \lambda = \underline{0.1}, s \in \left(0, \frac{2}{3}\right); \\ x'_{0.8}, & \text{if } \lambda = \underline{0.2}, s \in \left(0, \frac{1}{2}\right); \\ \underline{1}, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{C}_{\delta_2}(\lambda, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, s \in (0, 1); \\ x'_{0.9}, & \text{if } \lambda = \underline{0.1}, s \in \left(0, \frac{3}{4}\right); \\ x'_{0.8}, & \text{if } \lambda = \underline{0.2}, s \in \left(0, \frac{2}{3}\right); \\ \underline{1}, & \text{otherwise.} \end{cases}$$

Theorem 3.5 *Let $(X, \mathcal{C})$ be a fuzzy closure space. Define a function $\delta_{\mathcal{C}} : Pt(X) \times I^X \to I$ as follows:*

$$\delta_{\mathcal{C}}(x_r, \lambda) = \begin{cases} \bigwedge\{s \in I : \mathcal{C}(\lambda, s') \leq x'_r\}, & \text{if } \mathcal{C}(\lambda, s') \leq x'_r; \\ 1, & \text{if } \mathcal{C}(\lambda, s') > x'_r. \end{cases}$$

Then $\delta_{\mathcal{C}}$ is a fuzzy $k_\circ$-preproximity on X.

Proof

(K1) Clear.

(K2) Let $\lambda \leq \mu$, $r \leq t$ and $q \leq s$, then $\mathcal{C}(\lambda, s') \leq \mathcal{C}(\mu, q')$. Then, we have three cases:
  (a) If $\mathcal{C}(\lambda, s') > x'_r$, then $\delta_{\mathcal{C}}(x_r, \lambda) = 1$. But $\mathcal{C}(\lambda, s') \leq \mathcal{C}(\mu, q')$. Therefore, $\delta_{\mathcal{C}}(x_t, \mu) = 1$.
  (b) If $\mathcal{C}(\mu, q') > x'_t$, then $\delta_{\mathcal{C}}(x_t, \mu) = 1$. Therefore $\delta_{\mathcal{C}}(x_r, \lambda) \leq \delta_{\mathcal{C}}(x_t, \mu)$.
  (c) If $\mathcal{C}(\mu, q') \leq x'_t$, then $\mathcal{C}(\lambda, s') \leq x'_r$ and hence

$$\begin{aligned} \delta_{\mathcal{C}}(x_r, \lambda) &= \bigwedge\{s \in I : \mathcal{C}(\lambda, s') \leq x'_r\} \\ &\leq \bigwedge\{q \in I : \mathcal{C}(\mu, q') \leq x'_t\} \\ &= \delta_{\mathcal{C}}(x_t, \mu). \end{aligned}$$

(K3) If $\delta_{\mathcal{C}}(x_r, \mu) \neq 1$, then there exists $1 \neq s \in I$ such that $\mu \leq \mathcal{C}(\mu, s') \leq x'_r$. Therefore $x_r \leq \mu'$, i.e., $x_r \in \mu'$.

(K4) Suppose that $\delta_{\mathcal{C}}(x_{r_1}, \rho_1) \vee \delta_{\mathcal{C}}(x_{r_2}, \rho_2) \ngeq \delta_{\mathcal{C}}(x_{r_1 \wedge r_2}, \rho_1 \vee \rho_2)$. Then there exists $t \in I$ such that $\delta_{\mathcal{C}}(x_{r_1}, \rho_1) \vee \delta_{\mathcal{C}}(x_{r_2}, \rho_2) < t < \delta_{\mathcal{C}}(x_{r_1 \wedge r_2}, \rho_1 \vee \rho_2)$. So, there exists q_i such that $\delta_{\mathcal{C}}(x_{r_i}, \rho_i) \leq q_i < t$ and $\mathcal{C}(\rho_i, q'_i) \leq x'_{r_i}$ for each $i = 1, 2$. Let $q = \bigvee q_i$. Then using (C3), we have $\mathcal{C}(\rho_i, q') \leq x'_{r_i}$, and using (C4),

$$\mathcal{C}(\rho_1 \vee \rho_2, q') = \mathcal{C}(\rho_1, q') \vee \mathcal{C}(\rho_2, q').$$

So, $\mathcal{C}(\rho_1 \vee \rho_2, q') \leq x'_{r_1 \wedge r_2}$. Then, $\delta_{\mathcal{C}}(x_{r_1 \wedge r_2}, \rho_1 \vee \rho_2) \leq q < t$. It is a contradiction. Thus $\delta_{\mathcal{C}}(x_{r_1}, \rho_1) \vee \delta_{\mathcal{C}}(x_{r_2}, \rho_2) \geq \delta_{\mathcal{C}}(x_{r_1 \wedge r_2}, \rho_1 \vee \rho_2)$.

□

Example 3.6 Define $\mathcal{C}_{\delta_1}$ and $\mathcal{C}_{\delta_2}$ as in Example 3.4. We will create fuzzy $k_\circ$-preproximities from the fuzzy closure operators as follows:

$$\delta_{\mathcal{C}_{\delta_1}}(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\delta_{\mathcal{C}_{\delta_2}}(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{4}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ 1, & \text{otherwise.} \end{cases}$$

Example 3.7 Let X be a set. Define the function $\mathcal{C} : I^X \times I_0 \to I^X$ as follows:

$$\mathcal{C}(\lambda, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, s \in I_0; \\ \wedge x'_r, & \text{if } x_r \in \lambda', s \in I_0; \\ \underline{1}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{C}$ is a fuzzy closure operator on X. We will create a fuzzy $k_\circ$-preproximity from the fuzzy closure operator as follows:

$$\delta_{\mathcal{C}}(x_r, \lambda) = \begin{cases} 0, & \text{if } x_r \in \lambda'; \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.8 *Let (X, δ) be a fuzzy $k_\circ$-preproximity space. Then $\delta_{\mathcal{C}_\delta} \leq \delta$.*

Proof Suppose that $\delta_{\mathcal{C}_\delta} \not\leq \delta$. Then there exists $x_r \in Pt(X)$ and $\lambda \in I^X$ such that $\delta_{\mathcal{C}_\delta}(x_r, \lambda) \not\leq \delta(x_r, \lambda)$. Hence, there exists $s \in I$, such that $\delta_{\mathcal{C}_\delta}(x_r, \lambda) > s > \delta(x_r, \lambda)$. Using the definition of $\mathcal{C}_\delta$, we have $\mathcal{C}_\delta(\lambda, s') \leq x'_r$. From the definition of $\delta_{\mathcal{C}_\delta}$, it follows $\delta_{\mathcal{C}_\delta}(x_r, \lambda) \leq s$, which is a contradiction. □

4 Connection with fuzzy cotopology

In this section, we study the connection between fuzzy cotopology and fuzzy $k_\circ$-preproximity relation δ.

Theorem 4.1 *Let (X, δ) be a fuzzy $k_\circ$-preproximity space. Define the function $\mathcal{F}_{\mathcal{C}_\delta} : I^X \to I$, for each $x \in X$, by:*

$$\mathcal{F}_{\mathcal{C}_\delta}(\lambda) = \begin{cases} \bigvee\{s \in I : \mathcal{C}_\delta(\lambda, s) \leq x'_r\}, & \text{if } \mathcal{C}_\delta(\lambda, s) \leq x'_r; \\ 1, & \text{if } \mathcal{C}_\delta(\lambda, s) > x'_r. \end{cases}$$

Then $\mathcal{F}_{\mathcal{C}_\delta}$ is a fuzzy cotopology on X induced by $\mathcal{C}_\delta$.

Proof

(F1) Clear.

(F2) Suppose that $\mathcal{F}_{\mathcal{C}_\delta}(\lambda_1 \vee \lambda_2) \not\geq \mathcal{F}_{\mathcal{C}_\delta}(\lambda_1) \wedge \mathcal{F}_{\mathcal{C}_\delta}(\lambda_2)$. Then there exists $s \in I$ such that

$$\mathcal{F}_{\mathcal{C}_\delta}(\lambda_1 \vee \lambda_2) < s < \mathcal{F}_{\mathcal{C}_\delta}(\lambda_1) \wedge \mathcal{F}_{\mathcal{C}_\delta}(\lambda_2).$$

So, there exists q_i such that $s < q_i \leq \mathcal{F}_{\mathcal{C}_\delta}(\lambda_i)$ and $\mathcal{C}_\delta(\lambda_i, q_i) \leq x'_{r_i}$ for each $i = 1, 2$. Let $q = \bigwedge q_i$. Then using (C3), we have $\mathcal{C}_\delta(\lambda_i, q) \leq x'_{r_i}$, and using (C4),

$$\mathcal{C}_\delta(\lambda_1 \vee \lambda_2, q) = \mathcal{C}_\delta(\lambda_1, q) \vee \mathcal{C}_\delta(\lambda_2, q).$$

So, $\mathcal{C}_\delta(\lambda_1 \vee \lambda_2, q) \leq x'_{r_1} \vee x'_{r_2}$, i.e., $\mathcal{C}_\delta(\lambda_1 \vee \lambda_2, q) \leq x'_{r_1 \wedge r_2}$. Then, $s < q \leq \mathcal{F}_{\mathcal{C}_\delta}(\lambda_1 \vee \lambda_2)$. It is a contradiction. Thus $\mathcal{F}_{\mathcal{C}_\delta}(\lambda_1 \vee \lambda_2) \geq \mathcal{F}_{\mathcal{C}_\delta}(\lambda_1) \wedge \mathcal{F}_{\mathcal{C}_\delta}(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.

(F3) Let $\lambda = \bigwedge_{k\in\Gamma} \lambda_k$. Then $\mathcal{C}_\delta(\lambda, s) \leq \mathcal{C}_\delta(\lambda_k, s)$ for all $s \in I_0$ and $k \in \Gamma$. Suppose that $\mathcal{F}_{\mathcal{C}_\delta}(\lambda) \not\geq \bigwedge_{k\in\Gamma} \mathcal{F}_{\mathcal{C}_\delta}(\lambda_k)$. Then $\mathcal{F}_{\mathcal{C}_\delta}(\lambda) < \alpha < \bigwedge_{k\in\Gamma} \mathcal{F}_{\mathcal{C}_\delta}(\lambda_k)$ where $\alpha \in I$. Hence, there exists $q_k \in I$ with $\alpha < q_k \leq \mathcal{F}_{\mathcal{C}_\delta}(\lambda_k)$ for all $k \in \Gamma$ such that $\mathcal{C}_\delta(\lambda_k, q_k) \leq x'_{r_k}$ for all $k \in \Gamma$. Let $q = \bigwedge_{k\in\Gamma} q_k$. Then using (C3), we have $\mathcal{C}_\delta(\lambda_k, q) \leq x'_{r_k}$ for all $k \in \Gamma$. So, $\mathcal{C}_\delta(\lambda, q) \leq \mathcal{C}_\delta(\lambda_k, q) \leq x'_{r_k}$ for all $k \in \Gamma$, i.e., $\mathcal{C}_\delta(\lambda, q) \leq \bigvee_{k\in\Gamma} x'_{r_k}$. Putting $\bigvee_{k\in\Gamma} x'_{r_k} = x'_r$. Then, $\mathcal{C}_\delta(\lambda, q) \leq x'_r$. So, $\mathcal{F}_{\mathcal{C}_\delta}(\lambda) \geq q > \alpha$, it is a contradiction. Thus, $\mathcal{F}_{\mathcal{C}_\delta}(\bigwedge_{k\in\Gamma} \lambda_k) \geq \bigwedge_{k\in\Gamma} \mathcal{F}_{\mathcal{C}_\delta}(\lambda_k)$.

□

Example 4.2 From Example 3.4, we have

$$\mathcal{F}_{\mathcal{C}_{\delta_1}} = \begin{cases} \frac{1}{3}, & \text{if } \lambda = \underline{0.1}; \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{F}_{\mathcal{C}_{\delta_2}}(\lambda) = \begin{cases} \frac{1}{4}, & \text{if } \lambda = \underline{0.1}; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.2}; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that both $\mathcal{F}_{\mathcal{C}_{\delta_1}}, \mathcal{F}_{\mathcal{C}_{\delta_2}} : I^X \to I$ satisfies the conditions (F1), (F2) and (F3).

Theorem 4.3 *Let (X, δ) be a principal fuzzy $k_\circ$-preproximity space. Define the function $\mathcal{F}_\delta : I^X \to I$, for each $x \in X$, by*

$$\mathcal{F}_\delta(\lambda) = \begin{cases} \bigvee \delta(x_r, \lambda')' & \text{if } x_r \in \lambda; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}_\delta$ is a fuzzy cotopology on X.

Proof

(F1) Clear.
(F2)

$$\begin{aligned} \mathcal{F}_\delta(\lambda) \wedge \mathcal{F}_\delta(\mu) &= \left(\bigvee_{x_r\in\lambda} \delta(x_r, \lambda')' \right) \wedge \left(\bigvee_{x_t\in\mu} \delta(x_t, \mu')' \right) \\ &= \bigvee_{x_r\in\lambda, x_t\in\mu} (\delta(x_r, \lambda')' \wedge \delta(x_t, \mu')') \\ &= \bigvee_{x_r\in\lambda, x_t\in\mu} (\delta(x_r, \lambda') \vee \delta(x_t, \mu'))' \\ &\leq \bigvee_{x_k\in\lambda\vee\mu} \delta(x_k, \lambda' \wedge \mu')' \\ &= \bigvee_{x_k\in\lambda\vee\mu} \delta(x_k, (\lambda \vee \mu)')' \\ &= \mathcal{F}_\delta(\lambda \vee \mu). \end{aligned}$$

(F3) If $\{\lambda_i : i \in \Gamma\} \subset I^X$, then we have

$$\begin{aligned}
\bigwedge_{i\in\Gamma}\mathcal{F}_\delta(\lambda_i) &= \bigwedge_{i\in\Gamma}\left(\bigvee_{x_r\in\lambda_i}\delta(x_r,\lambda_i')'\right)\\
&= \bigvee_{x_r\in\lambda_i}\left(\bigvee_{i\in\Gamma}\delta(x_r,\lambda_i')\right)'\\
&\le \bigvee_{x_r\in\bigwedge_{i\in\Gamma}\lambda_i}\delta\left(x_r,\bigvee_{i\in\Gamma}\lambda_i'\right)'\\
&= \bigvee_{x_r\in\bigwedge_{i\in\Gamma}\lambda_i}\delta\left(x_r,\left(\bigwedge_{i\in\Gamma}\lambda_i\right)'\right)' \qquad \text{(since } \delta \text{ is principal)}\\
&= \mathcal{F}_\delta\left(\bigwedge_{i\in\Gamma}\lambda_i\right).
\end{aligned}$$

□

Example 4.4 Define $\delta : Pt(X)\times I^X \to I$ as in Example 3.2. It is clear that (X,δ) is a principal preproximity space. Thus we can define a fuzzy cotopology $\mathcal{F}_\delta$ on X as:

$$\mathcal{F}_\delta(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \text{ or } \lambda = \underline{1};\\ \frac{3}{4}, & \text{if } \lambda = \underline{0.9};\\ \frac{2}{3}, & \text{if } \lambda = \underline{0.8};\\ \frac{1}{2}, & \text{if } \lambda = \underline{0.7};\\ \frac{1}{4}, & \text{if } \lambda = \underline{0.6};\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 4.5 *Let $\mathcal{F}$ be a fuzzy cotopology on X. Define the function $\delta_{\mathcal{F}} : Pt(X)\times I^X \to I$ as follows:*

$$\delta_{\mathcal{F}}(x_r,\lambda) = \begin{cases} \bigwedge\{\mathcal{F}(\gamma)' : \gamma \in \phi_{x_r,\lambda}\}, & \text{if } \phi_{x_r,\lambda} \ne \underline{0};\\ 1, & \text{if } \phi_{x_r,\lambda} = \underline{0}.\end{cases}$$

where $\phi_{x_r,\lambda} = \{\gamma \in L^X : x_r \le \gamma \le \lambda'\}$. Then:

(1) *$\delta_{\mathcal{F}}$ is a fuzzy $k_\circ$-preproximity on X.*
(2) *$\delta_{\mathcal{F}}$ is principal.*
(3) *If δ is a principal fuzzy $k_\circ$-preproximity on X, then $\delta_{\mathcal{F}_\delta} \le \delta$.*

Proof

(1)
(K1) $\delta_{\mathcal{F}}(x_r,\underline{0}) = \bigwedge\{\mathcal{F}(\gamma)' : \gamma \in \phi_{x_r,\underline{0}}\}$, $\phi_{x_r,\underline{0}} = \{\gamma \in I^X : x_r \le \gamma \le \underline{1}\}$. Since $\mathcal{F}(\underline{1})' = 0$. Thus $\delta_{\mathcal{F}}(x_r,\underline{0}) = 0$ for all $r \in I_0$.
(K2) Let $\lambda \le \mu$ and $r \le t$. Then $\phi_{x_r,\lambda} = \{\gamma \in I^X : x_r \le \gamma \le \lambda'\} \supseteq \{\gamma \in I^X : x_t \le \gamma \le \mu'\} = \phi_{x_t,\mu}$. Now, we have three cases:
(a) If $\phi_{x_r,\lambda} = \underline{0}$, then $\phi_{x_t,\mu} = \underline{0}$ with $\phi_{x_r,\lambda} \supseteq \phi_{x_t,\mu}$. Hence $\delta_{\mathcal{F}}(x_r,\lambda) = 1$ and $\delta_{\mathcal{F}}(x_t,\mu) = 1$, i.e., $\delta_{\mathcal{F}}(x_r,\lambda) = \delta_{\mathcal{F}}(x_t,\mu)$.
(b) If $\phi_{x_t,\mu} = \underline{0}$, then $\delta_{\mathcal{F}}(x_t,\mu) = 1$. Therefore $\delta_{\mathcal{F}}(x_t,\mu) \ge \delta_{\mathcal{F}}(x_r,\lambda)$.
(c) If $\phi_{x_t,\mu} \ne \underline{0}$. Since $\phi_{x_r,\lambda} \supseteq \phi_{x_t,\mu}$, then $\phi_{x_r,\lambda} \ne \underline{0}$ and hence

$$\begin{aligned}
\delta_{\mathcal{F}}(x_r,\lambda) &= \bigwedge\{\mathcal{F}(\gamma)' : \gamma \in \phi_{x_r,\lambda}\}\\
&\le \bigwedge\{\mathcal{F}(\gamma)' : \gamma \in \phi_{x_t,\mu}\}\\
&= \delta_{\mathcal{F}}(x_t,\mu).
\end{aligned}$$

(K3) If $\delta_{\mathcal{F}}(x_r, \mu) \neq 1$, then $\phi_{x_r,\mu} \neq \underline{0}$ and hence $x_r \in \mu'$.

(K4) If $\phi_{x_{r_1},\lambda_1} = \underline{0}$ or $\phi_{x_{r_2},\lambda_2} = \underline{0}$, it is trivial. Let $\phi_{x_{r_1},\lambda_1} \neq \underline{0}$ and $\phi_{x_{r_2},\lambda_2} \neq \underline{0}$ and let $\delta_{\mathcal{F}}(x_{r_1}, \lambda_1) \vee \delta_{\mathcal{F}}(x_{r_2}, \lambda_2) \not\geq \delta_{\mathcal{F}}(x_{r_1 \wedge r_2}, \lambda_1 \vee \lambda_2)$. Then there exists $t \in I$ such that $\delta_{\mathcal{F}}(x_{r_1}, \lambda_1) \vee \delta_{\mathcal{F}}(x_{r_2}, \lambda_2) < t < \delta_{\mathcal{F}}(x_{r_1 \wedge r_2}, \lambda_1 \vee \lambda_2)$. So, there exists q_i such that $\delta_{\mathcal{F}}(x_{r_i}, \lambda_i) \leq q_i < t$ and $q_i = \mathcal{F}(\gamma_i)'$, $x_{r_i} \leq \gamma_i \leq \lambda_i'$ for each $i = 1, 2$. Let $q = \bigvee q_i$. Then $q = q_1$ or $q = q_2$. Let $q = q_1$. Then $\delta_{\mathcal{F}}(x_{r_i}, \lambda_i) \leq q$. Put $\gamma = \gamma_1$, we have $x_{r_1 \wedge r_2} \leq \gamma \leq \lambda_1' \wedge \lambda_2' = (\lambda_1 \vee \lambda_2)'$. Thus $\gamma \in \phi_{x_{r_1 \wedge r_2},\lambda_1 \vee \lambda_2}$ and hence $\delta_{\mathcal{F}}(x_{r_1 \wedge r_2}, \lambda_1 \vee \lambda_2) \leq \mathcal{F}(\gamma)' < t$. It is a contradiction. Thus $\delta_{\mathcal{F}}(x_{r_1}, \lambda_1) \vee \delta_{\mathcal{F}}(x_{r_2}, \lambda_2) \geq \delta_{\mathcal{F}}(x_{r_1 \wedge r_2}, \lambda_1 \vee \lambda_2)$.

(2) (K5) For each $\nu_i \in I^X$ with $x_r \leq \nu_i \leq \lambda_i'$, we have $x_r \leq \bigwedge_{i\in\Gamma} \nu_i \leq \bigwedge_{i\in\Gamma} \lambda_i'$. By the definition of $\delta_{\mathcal{F}}$ and fuzzy cotopology $\mathcal{F}$, we have

$$\delta_{\mathcal{F}}\left(x_r, \bigvee_{i\in\Gamma} \lambda_i\right) \leq \mathcal{F}\left(\bigwedge_{i\in\Gamma} \nu_i\right)' \leq \left(\bigwedge_{i\in\Gamma} \mathcal{F}(\nu_i)\right)'.$$

Hence,

$$\begin{aligned}
\bigvee_{i\in\Gamma} \delta_{\mathcal{F}}(x_r, \lambda_i) &= \bigvee_{i\in\Gamma} \left(\bigwedge \{\mathcal{F}(\nu_i)' : \nu_i \in \phi_{x_r,\lambda_i}\}\right) \\
&= \bigwedge \left(\bigvee_{i\in\Gamma} \{\mathcal{F}(\nu_i)' : \nu_i \in \phi_{x_r,\lambda_i}\}\right) \\
&= \bigwedge \left\{\left(\bigwedge_{i\in\Gamma} \mathcal{F}(\nu_i)\right)' : \nu_i \in \phi_{x_r,\lambda_i}\right\} \\
&\geq \bigwedge \left\{\mathcal{F}\left(\bigwedge_{i\in\Gamma} \nu_i\right)' : \bigwedge_{i\in\Gamma} \nu_i \in \phi_{x_r,\vee_{i\in\Gamma}\lambda_i}\right\} \\
&= \delta_{\mathcal{F}}\left(x_r, \bigvee_{i\in\Gamma} \lambda_i\right).
\end{aligned}$$

(3) Suppose that $\delta_{\mathcal{F}_\delta} \not\leq \delta$. Then there exists $x_r \in Pt(X)$, $\lambda \in I^X$ and $s \in I$ such that $\delta_{\mathcal{F}_\delta}(x_r, \lambda) > s > \delta(x_r, \lambda)$. From the definition of $\mathcal{F}_\delta$, we have $\mathcal{F}_\delta(\lambda') \geq \delta(x_r, \lambda)'$. Hence, $\mathcal{F}_\delta(\lambda')' \leq \delta(x_r, \lambda) < s$. Using the definition of $\delta_{\mathcal{F}}$, we have $\delta_{\mathcal{F}_\delta}(x_r, \lambda) \leq \mathcal{F}_\delta(\lambda')' < s$ which is a contradiction. Thus $\delta_{\mathcal{F}_\delta} \leq \delta$. □

Example 4.6 Consider $\mathcal{F}_\delta$ defined as in Example 4.4. We can obtain a fuzzy $k_\circ$-preproximity $\delta_{\mathcal{F}_\delta}$ on X as follows:

$$\delta_{\mathcal{F}_\delta}(x_r, \lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, r \in (0, 1]; \\ \frac{1}{4}, & \text{if } \lambda = \underline{0.1}, r \in (0, 0.9]; \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.2}, r \in (0, 0.8]; \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.3}, r \in (0, 0.7]; \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.4}, r \in (0, 0.6]; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $\delta_{\mathcal{F}_\delta}$ is a principal fuzzy $k_\circ$-preproximity.

5 Fuzzy $k_\circ$-preproximal neighborhood

In this section, we introduce the concept of the fuzzy neighborhood system $\mathcal{N}_\delta$ which is induced by a fuzzy $k_\circ$-preproximity relation δ.

Definition 5.1 Let X be a set. A map $\mathcal{N} : X \times I^X \times I \to I$ is called a fuzzy neighborhood system iff $\mathcal{N}$ satisfies the following axioms:

(N0) $\mathcal{N}(x, \underline{1}, s) = 1$.

(N1) $\mathcal{N}(x, \lambda, s) \leq \lambda(x)$.
(N2) If $\lambda \leq \mu$, then $\mathcal{N}(x, \lambda, s) \leq \mathcal{N}(x, \mu, s)$.
(N3) If $s \leq q$, then $\mathcal{N}(x, \lambda, q) \leq \mathcal{N}(x, \lambda, s)$.
(N4) $\mathcal{N}(x, \lambda, s) \wedge \mathcal{N}(x, \mu, s) = \mathcal{N}(x, \lambda \wedge \mu, s)$.

Theorem 5.2 *Let (X, δ) be a fuzzy $k_\circ$-preproximity space. Define the map $\mathcal{N}_\delta : X \times I^X \times I \to I$ by:*

$$\mathcal{N}_\delta(x, \lambda, s) = \begin{cases} \bigvee\{r : \delta(x_r, \lambda') < s'\}, & \text{if } \delta(x_r, \lambda') < s'; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{N}_\delta$ is a fuzzy neighborhood system, we call it a fuzzy $k_\circ$-preproximal neighborhood.

Proof

(N0) Since $\delta(x_r, \underline{0}) = 0$ for all $r \in I_0$. Then $\mathcal{N}_\delta(x, \underline{1}, s) = 1$.
(N1) Since $\delta(x_r, \lambda') < s'$. Then using (K3), we have $x_r \leq \lambda$. Thus $\mathcal{N}_\delta(x, \lambda, s) \leq \lambda(x)$.
(N2), (N3) Let $\lambda \leq \mu$ and $s \leq q$. Then $\delta(x_r, \mu') \leq \delta(x_r, \lambda') < q' \leq s'$. If $\delta(x_r, \lambda') \not< q'$ or $\delta(x_r, \mu') \not< s'$, it is trivial. Let $\delta(x_r, \lambda') < q'$ and $\delta(x_r, \mu') < s'$. Then, we have $\mathcal{N}_\delta(x, \lambda, q) \leq \mathcal{N}_\delta(x, \mu, s)$.
(N4)

$$\begin{aligned} \mathcal{N}_\delta(x, \lambda_1, s) \wedge \mathcal{N}_\delta(x, \lambda_2, s) &= \left(\bigvee\{r_1 : \delta(x_{r_1}, \lambda_1') < s'\}\right) \wedge \left(\bigvee\{r_2 : \delta(x_{r_2}, \lambda_2') < s'\}\right) \\ &= \bigvee\{r_1 \wedge r_2 : \delta(x_{r_1}, \lambda_1') < s', \quad \delta(x_{r_2}, \lambda_2') < s'\} \\ &\leq \bigvee\{r : \delta(x_r, (\lambda_1 \wedge \lambda_2)') < s'\} \\ &= \mathcal{N}_\delta(x, \lambda_1 \wedge \lambda_2, s). \end{aligned}$$

□

Example 5.3 Consider the fuzzy $k_\circ$-preproximity δ defined as in Example 3.2. Then we can induce a fuzzy neighborhood system $\mathcal{N}_\delta : X \times I^X \times I \to I$ from δ as:

$$\mathcal{N}_\delta(x, \lambda, s) = \begin{cases} 1, & \text{if } \lambda = \underline{1}, s \in (0, 1) \text{ and } x \in X; \\ 0.9, & \text{if } \lambda = \underline{0.9}, s \in \left(0, \frac{3}{4}\right) \text{ and } x_r \in \lambda, r \in (0, 0.9); \\ 0.8, & \text{if } \lambda = \underline{0.8}, s \in \left(0, \frac{2}{3}\right) \text{ and } x_r \in \lambda, r \in (0, 0.8); \\ 0.7, & \text{if } \lambda = \underline{0.7}, s \in \left(0, \frac{1}{2}\right) \text{ and } x_r \in \lambda, r \in (0, 0.7); \\ 0.6, & \text{if } \lambda = \underline{0.6}, s \in \left(0, \frac{1}{4}\right) \text{ and } x_r \in \lambda, r \in (0, 0.6); \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.4 Let $\delta_{\mathcal{C}}$ be the fuzzy $k_\circ$-preproximity defined as in Example 3.7. Then we can induce a fuzzy neighborhood system $\mathcal{N}_{\delta_{\mathcal{C}}} : X \times I^X \times I \to I$ from $\delta_{\mathcal{C}}$ as:

$$\mathcal{N}_{\delta_{\mathcal{C}}}(x, \lambda, s) = \begin{cases} \vee r, & \text{if } x_r \in \lambda, s \in I_1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.5 *Let $\mathcal{N}_\delta$ be a fuzzy $k_\circ$-preproximal neighborhood induced by a fuzzy $k_\circ$-preproximity δ. Define the function $\mathcal{C}_{\mathcal{N}_\delta} : I^X \times I_0 \to I^X$ by*

$$[\mathcal{C}_{\mathcal{N}_\delta}(\lambda, s)](x) = \mathcal{N}_\delta(x, \lambda', s)'.$$

Then $(X, \mathcal{C}_{\mathcal{N}_\delta})$ is a fuzzy closure space.

Proof

(C1) Since $[\mathcal{C}_{\mathcal{N}_\delta}(\underline{0}, s)](x) = \mathcal{N}_\delta(x, \underline{1}, s)' = 0$ for each $x \in X$. Then $\mathcal{C}_{\mathcal{N}_\delta}(\underline{0}, s) = \underline{0}$.
(C2) Since $\mathcal{N}_\delta(x, \lambda', s) \leq \lambda'(x)$. Then $\lambda(x) \leq \mathcal{N}_\delta(x, \lambda', s)' = [\mathcal{C}_{\mathcal{N}_\delta}(\lambda, s)](x)$ for each $x \in X$. Thus $\lambda \leq \mathcal{C}_{\mathcal{N}_\delta}(\lambda, s)$.
(C3) If $\lambda \leq \mu$ and $s \leq q$, then $\mathcal{N}_\delta(x, \mu', q) \leq \mathcal{N}_\delta(x, \lambda', s)$, i.e., $[\mathcal{C}_{\mathcal{N}_\delta}(\lambda, s)](x) \leq [\mathcal{C}_{\mathcal{N}_\delta}(\mu, q)](x)$ for each $x \in X$. Thus $\mathcal{C}_{\mathcal{N}_\delta}(\lambda, s) \leq \mathcal{C}_{\mathcal{N}_\delta}(\mu, q)$.

(C4) For each $x \in X$, we have

$$\begin{aligned}[\mathcal{C}_{\mathcal{N}_\delta}(\lambda \vee \mu, s)](x) &= \mathcal{N}_\delta(x, (\lambda \vee \mu)', s)' = \mathcal{N}_\delta(x, \lambda' \wedge \mu', s)' \\ &= (\mathcal{N}_\delta(x, \lambda', s) \wedge \mathcal{N}_\delta(x, \mu', s))' = \mathcal{N}_\delta(x, \lambda', s)' \vee \mathcal{N}_\delta(x, \mu', s)' \\ &= [\mathcal{C}_{\mathcal{N}_\delta}(\lambda, s)](x) \vee [\mathcal{C}_{\mathcal{N}_\delta}(\mu, s)](x).\end{aligned}$$

Thus $\mathcal{C}_{\mathcal{N}_\delta}(\lambda \vee \mu, s) = \mathcal{C}_{\mathcal{N}_\delta}(\lambda, s) \vee \mathcal{C}_{\mathcal{N}_\delta}(\mu, s)$. □

Example 5.6 Consider the fuzzy neighborhood system $\mathcal{N}_\delta$ defined as in Example 5.3. Then we can induce a fuzzy closure operator $\mathcal{C}_{\mathcal{N}_\delta} : I^X \times I_0 \to I^X$ as follows:

$$\mathcal{C}_{\mathcal{N}_\delta}[(\lambda, s)](x) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, s \in (0,1) \text{ and } x \in X; \\ 0.1, & \text{if } \lambda = \underline{0.1}, s \in \left(0, \frac{3}{4}\right) \text{ and } x_r \in \lambda', r \in (0, 0.9); \\ 0.2, & \text{if } \lambda = \underline{0.2}, s \in \left(0, \frac{2}{3}\right) \text{ and } x_r \in \lambda', r \in (0, 0.8); \\ 0.3, & \text{if } \lambda = \underline{0.3}, s \in \left(0, \frac{1}{2}\right) \text{ and } x_r \in \lambda', r \in (0, 0.7); \\ 0.4, & \text{if } \lambda = \underline{0.4}, s \in \left(0, \frac{1}{4}\right) \text{ and } x_r \in \lambda', r \in (0, 0.6); \\ 1, & \text{otherwise.} \end{cases}$$

Example 5.7 Let $\delta_{\mathcal{C}}$ be the fuzzy $k_\circ$-preproximity defined as in Example 3.7. Then we can induce a fuzzy closure operator $\mathcal{C}_{\mathcal{N}_{\delta_{\mathcal{C}}}} : I^X \times I_0 \to I^X$ from $\mathcal{N}_{\delta_{\mathcal{C}}}$ as:

$$\mathcal{C}_{\mathcal{N}_{\delta_{\mathcal{C}}}}(x, \lambda, s) = \begin{cases} \wedge r', & \text{if } x_r \in \lambda', s \in I_1; \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 5.8 *Let (X, δ) be a fuzzy $k_\circ$-preproximity space. Define the function $\mathcal{N}_{\mathcal{C}_\delta} : X \times I^X \times I \to I$ by:*

$$\mathcal{N}_{\mathcal{C}_\delta}(x, \lambda, s) = [\mathcal{C}_\delta(\lambda', s)]'(x)$$

for each $x \in X$. Then $\mathcal{N}_{\mathcal{C}_\delta}$ is a fuzzy $k_\circ$-preproximal neighborhood.

Proof For each $x \in X$, we have

$$\begin{aligned}\mathcal{N}_{\mathcal{C}_\delta}(x, \lambda, s) &= [\mathcal{C}_\delta(\lambda', s)]'(x) \\ &= \left(\bigwedge \{x_r' \in I^X : \delta(x_r, \lambda') < s'\}\right)'(x) \\ &= \left(\bigvee \{x_r : \delta(x_r, \lambda') < s'\}\right)(x) \\ &= \bigvee \{r : \delta(x_r, \lambda') < s'\} \\ &= \mathcal{N}_\delta(x, \lambda, s).\end{aligned}$$

□

Example 5.9 Let δ be the fuzzy $k_\circ$-preproximity defined as in Example 3.2. We can obtain a fuzzy closure operator $\mathcal{C}_\delta : I^X \times I_0 \to I^X$ as follows:

$$\mathcal{C}_\delta(\lambda, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, s \in (0,1); \\ x'_{0.9}, & \text{if } \lambda = \underline{0.1}, s \in \left(0, \frac{3}{4}\right); \\ x'_{0.8} & \text{if } \lambda = \underline{0.2}, s \in \left(0, \frac{2}{3}\right); \\ x'_{0.7}, & \text{if } \lambda = \underline{0.3}, s \in \left(0, \frac{1}{2}\right); \\ x'_{0.6}, & \text{if } \lambda = \underline{0.4}, s \in \left(0, \frac{1}{4}\right); \\ \underline{1}, & \text{otherwise.} \end{cases}$$

We can obtain a fuzzy neighborhood system $\mathcal{N}_{\mathcal{C}_\delta} : X \times I^X \times I \to I$ as follows:

$$\mathcal{N}_{\mathcal{C}_\delta}(x, \lambda, s) = \begin{cases} 1, & \text{if } \lambda = \underline{1}, s \in (0, 1) \text{ and } x \in X; \\ 0.9, & \text{if } \lambda = \underline{0.9}, s \in \left(0, \frac{3}{4}\right) \text{ and } x_r \in \lambda, r \in (0, 0.9); \\ 0.8, & \text{if } \lambda = \underline{0.8}, s \in \left(0, \frac{2}{3}\right) \text{ and } x_r \in \lambda, r \in (0, 0.8); \\ 0.7, & \text{if } \lambda = \underline{0.7}, s \in \left(0, \frac{1}{2}\right) \text{ and } x_r \in \lambda, r \in (0, 0.7); \\ 0.6, & \text{if } \lambda = \underline{0.6}, s \in \left(0, \frac{1}{4}\right) \text{ and } x_r \in \lambda, r \in (0, 0.6); \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 5.10 *Let $\mathcal{N}_\delta$ be a fuzzy $k_\circ$-preproximal neighborhood. Then the function $\tau_{\mathcal{C}_{\mathcal{N}\delta}} : I^X \to I$ defined by:*

$$\mathcal{T}_{\mathcal{C}_{\mathcal{N}_\delta}}(\lambda) = \bigvee \{s \in I : \mathcal{C}_{\mathcal{N}_\delta}(\lambda', s) = \lambda'\}$$

is a fuzzy topology.

6 Conclusions

Through this paper, we introduced the concepts of fuzzy $k_\circ$-preproximity and fuzzy neighborhood system. We investigated some of their properties. Also, we studied several relationships between fuzzy $k_\circ$-preproximity, fuzzy closure space and fuzzy neighborhood system. Moreover, we introduced many examples which supported these relations. Hence, further research may be undertaken towards this direction. That is, one may take further research to find the suitable way of defining the fuzzy $k_\circ$-preproximity on an appropriate lattice. It may also lead to the new significant properties between them and fuzzy closure spaces.

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